Since V is increasing on $[a,b]$,

in order to prove that $f \in R(V)$ on $[a,b]$ it is enough

to prove that 'f' satisfies Riemann's condition w.r.t

V on $[a,b]$

i.e.) To prove that,

For any $\epsilon > 0$ there exists a partition P of $[a,b]$

Such that for every $P \supseteq P_\epsilon$ and for every choice

of t_k, t_k' we have,

ETP:

$$0 < U(P,f,V) - L(P,f,V) < \epsilon$$

$$\text{i.e.) } \sum_{k=1}^{n} [M_k(f) - m_k(f)] \, |\Delta V_k| < \epsilon$$

$$\text{i.e.) } \sum_{k=1}^{n} [M_k(f) - m_k(f)] [\Delta V_k - |\Delta \alpha_k| + |\Delta \alpha_k|] < \epsilon$$

$[\because V \uparrow$ on $[a,b]$

To prove this we have to establish $|\Delta V_k| = \Delta V_k]$

the following inequalities, $\sum_{k=1} [M_k(f) - m_k(f)] [\Delta V_k - |\Delta \alpha_k| + \sum_{k=1}^{n} (M_k(f) - m_k(f)) |\Delta \alpha_k| < \epsilon$

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)] [\Delta V_k - [\Delta \alpha_k]] < \frac{\epsilon}{2} \longrightarrow (1)$$

$$\text{and } \sum_{k=1}^{n} [M_k(f) - m_k(f)] \, |\Delta \alpha_k| < \frac{\epsilon}{2} \longrightarrow (2)$$

Since f is bounded on $[a,b]$,

$\exists$ $M > 0$ such that $|f(x)| \le M$, $\forall x \in [a,b] \longrightarrow (3)$

Since $f \in R(\alpha)$ on $[a,b]$

For any given $\epsilon > 0$, $\exists$ a partition P_ϵ'

Such that, $\forall$ $P \supseteq P_\epsilon'$ and for all choices of points

t_k and t_k' in $[x_{k-1}, x_k]$ we have,

$|S(P,f,\alpha) - A| < \epsilon$

$$\left| \sum_{k=1}^{n} f(t_k) \Delta \alpha_k - \int_a^b f \, d\alpha \right| < \frac{\epsilon}{8}$$

$$\text{and } \left| \sum_{k=1}^{n} f(t_k') \Delta \alpha_k - \int_a^b f \, d\alpha \right| < \frac{\epsilon}{8}$$

Consider $\left| \sum (f(t_k) - f(t_k)) \Delta\alpha_k \right|$

Combining the two inequalities, we have,

Part II

$$= \left| \sum_{k=1}^{n} f(t_k)\Delta\alpha_k + \int_a^b f\,d\alpha - \int_a^b f\,d\alpha - \sum_{k=1}^{n} f(t'_k)\Delta\alpha_k \right|$$

$$\le \left| \sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha \right| + \left| \sum_{k=1}^{n} f(t'_k)\Delta\alpha_k - \int_a^b f\,d\alpha \right|$$

$$< \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{2\epsilon}{8} = \frac{\epsilon}{4}$$

$$\therefore \left| \sum [f(t_k) - f(t'_k)]\Delta\alpha_k \right| < \frac{\epsilon}{4} \longrightarrow (4)$$

Consider,

$$V(b) = V_\alpha(a,b) = \sup\left\{ \sum_{k=1}^{n} |\Delta\alpha_k| : P \in \mathcal{P}[a,b] \right\}$$

$\therefore V(b) - \frac{\epsilon}{4M}$ is not an upper bounde of the set

$\left\{ \sum_{k=1}^{n} |\Delta\alpha_k| : P \in \mathcal{P}[a,b] \right\}$

$\Rightarrow \exists$ a partition P''_ϵ of $[a,b]$ such that

$$V(b) - \frac{\epsilon}{4M} < \sum_{k=1}^{n} |\Delta\alpha_k|$$

ie) $\cancel{\forall\ P \supseteq P''_\epsilon}$,

$$V(b) - \sum_{k=1}^{n} |\Delta\alpha_k| < \frac{\epsilon}{4M} \longrightarrow (5)$$

Let $P_\epsilon = P'_\epsilon \cup P''_\epsilon$

Then for $P \supseteq P_\epsilon$, we have $P \supseteq P'_\epsilon$ & $P \supseteq P''_\epsilon$

For such 'p', inequalities (4) and (5) hold good.

$\therefore$ For every $P \supseteq P_\epsilon$ and for every choice of t_k & t'_k in $[x_{k-1}, x_k]$ we have,

$$\sum_{k=1}^{n} (f(t_k) - f(t'_k))\Delta\alpha_k < \frac{\epsilon}{4} \longrightarrow (6)$$

and $$V(b) - \sum_{k=1}^{n} |\Delta\alpha_k| < \frac{\epsilon}{4M} \longrightarrow (7)$$

To establish (1)

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consider. $\Delta V_k = V(x_k) - V(x_{k-1})$

$= V_a(a, x_k) - V_a(a, x_{k-1})$

$= V_a(x_{k-1}, x_k)$

$= \sup\{\sum |\Delta \alpha_k| : P \in \mathcal{P}[x_{k-1}, x_k]\}$

$\Delta V_k \geq |\Delta \alpha_k|, \ \forall k$

ie) $\Delta V_k - |\Delta \alpha_k| \geq 0, \ \forall k$

Further, $\forall k$

$M_k(f) \leq M$

$m_k(f) \leq M$ By (3)

$\therefore M_k(f) - m_k(f) \leq |M_k(f) - m_k(f)|$

$\leq |M_k(f)| + |m_k(f)|$

$\leq M + M = 2M$

$\therefore M_k(f) - m_k(f) \leq 2M$

consider

$$\sum_{k=1}^{n} [M_k(f) - m_k(f)][\Delta V_k - |\Delta \alpha_k|] \leq \sum_{k=1}^{n} 2M\{\Delta V_k - |\Delta \alpha_k|\}$$

$$= 2M\left\{\sum_{k=1}^{n} \Delta V_k - \sum_{k=1}^{n} |\Delta \alpha_k|\right\}$$

$$\Rightarrow \sum_{k=1}^{n} [M_k(f) - m_k(f)][\Delta V_k - |\Delta \alpha_k|] = 2M\left\{V(b) - V(a) - \sum_{k=1}^{n} |\Delta \alpha_k|\right\}$$

$V(b) = \sup\{\sum \Delta \alpha_k : P \in \mathcal{P}(x_{k-1}]\}$ $= 2M\left\{V(b) - \sum_{k=1}^{n} |\Delta \alpha_k|\right\}$

$< 2M \cdot \frac{\epsilon}{4M}$ [By (4)] $\because [V(a) = 0]$

$= \frac{\epsilon}{2}$

$$\Rightarrow \sum_{k=1}^{n} [M_k(f) - m_k(f)][\Delta V_k - |\Delta \alpha_k|] < \frac{\epsilon}{2} \longrightarrow (I)$$

To establish (2)

Let $A(P) = \{k : \Delta \alpha_k \geq 0\}$

and $B(P) = \{k : \Delta \alpha_k \leq 0\}$

ie) $k = 1, 2, 3 \ldots n$ is the member of the $A(P)$

if $\alpha(x_k) - \alpha(x_{k-1}) > 0$

$|||^{ly}$, k is the member of the set $B(P)$

if $\alpha(x_k) - \alpha(x_{k-1}) < 0$

Hence $A(P) \cup B(P) = \{1, 2, 3 \ldots n\}$

Let $h = \dfrac{\epsilon}{4V(b)} > 0 \longrightarrow (B)$

If $k \in A(P)$ choose $t_k, t_k' \in [x_{k-1}, x_k]$

Such that $M_k(f) - m_k(f) - h < f(t_k) - f(t_k')$

ie) $M_k(f) - m_k(f) < f(t_k) - f(t_k') + h \longrightarrow (8)$

If $k \in B(P)$ choose $t_k', t_k \in [x_{k-1}, x_k]$

Such that $M_k(f) - m_k(f) - h < f(t_k') - f(t_k)$

ie) $M_k(f) - m_k(f) < f(t_k') - f(t_k) + h \longrightarrow (9)$

Now, $\sum_{k=1}^{n} (M_k(f) - m_k(f)) |\Delta\alpha_k|$

$$= \sum_{k \in A(P)} (M_k(f) - m_k(f)) |\Delta\alpha_k| + \sum_{k \in B(P)} (M_k(f) - m_k(f)) |\Delta\alpha_k|$$

By using (8) and (9)

$$\sum_{k=1}^{n} (M_k(f) - m_k(f)) |\Delta\alpha_k| < \sum_{k \in A(P)} \{f(t_k) - f(t_k') + h\} |\Delta\alpha_k|$$

$$+ \sum_{k \in B(P)} \{f(t_k') - f(t_k) + h\} |\Delta\alpha_k|$$

$$= \sum_{k \in A(P)} \{f(t_k) - f(t_k')\} |\Delta\alpha_k| + \sum_{k \in B(P)} \{f(t_k') - f(t_k)\} |\Delta\alpha_k|$$

$$+ h \sum_{k \in A(P)} |\Delta\alpha_k| + h \sum_{k \in B(P)} |\Delta\alpha_k|$$

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$$= \sum_{k \in A(P)} \{ f(t_k) - f(t_k') \} \Delta\alpha_k + \sum_{k \in B(P)} \{ f(t_k') - f(t_k) \} (-\Delta\alpha_k)$$

$$+ h \sum_{k \in A(P)} |\Delta\alpha_k| + h \sum_{k \in B(P)} |\Delta\alpha_k|$$

$\{ \because |\Delta\alpha_k| \geq 0 \text{ for } k \in A(P)$

$$= \sum_{k \in A(P)} \{ f(t_k) - f(t_k') \} \Delta\alpha_k + \sum_{k \in B(P)} \{ f(t_k) - f(t_k') \} \Delta\alpha_k$$

$|\Delta\alpha_k| \leq 0 \text{ for } k \in B(P) \}$

$$+ h \sum_{k \in A(P)} |\Delta\alpha_k| + h \sum_{k \in B(P)} |\Delta\alpha_k|$$

$$= \sum_{k \in A(P) \cup B(P)} \{ f(t_k) - f(t_k') \} \Delta\alpha_k + h \sum_{k \in A(P) \cup B(P)} |\Delta\alpha_k|$$

$$= \sum_{k=1}^{n} \{ f(t_k) - f(t_k') \} \Delta\alpha_k + h \sum_{k=1}^{n} |\Delta\alpha_k|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4V(b)} V(b)$$

$\{ \because V(b) = \sup \{ \sum |\Delta\alpha_k| : P \in \mathcal{P} \}$

$\Rightarrow \sum |\Delta\alpha_k| \leq V(b) \}$

$$< \frac{\epsilon}{2}$$

$$\therefore \sum_{k=1}^{n} \{ M_k(f) - m_k(f) \} |\Delta\alpha_k| < \frac{\epsilon}{2} \longrightarrow ②$$

Combining ① and ② we have

$$\sum_{k=1}^{n} \{ M_k(f) - m_k(f) \} |\Delta V_k| < \epsilon$$

i.e.) $0 \leq U(P, f, V) - L(P, f, V) < \epsilon$

$\Rightarrow f \in R(V)$ on $[a, b]$

Theorem : 2

(X) V.V.I ★ Let α be of bounded variation on $[a, b]$ and assume that $f \in R(\alpha)$ on $[a, b]$.

Then $f \in R(\alpha)$ on every subinterval $[c, d]$ of $[a, b]$.

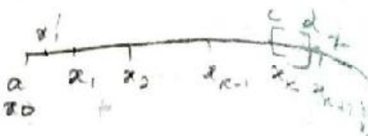
Proof:

Given: (i) α is of bounded variation on $[a,b]$

(ii) $f \in R(\alpha)$ on $[a,b]$

To prove: $f \in R(\alpha)$ on every subinterval $[c,d]$ of $[a,b]$.

Let $V(x)$ denote total variation of α on $[a,x]$ with $V(a)=0$.



i.e.) $V(x) = V_\alpha(a,x)$ if $a < x \le b$

and $V(a) = 0$

then, since α is of bounded variation on $[a,b]$,

$\alpha = V - (V - \alpha)$ where both V and $V-\alpha$ are increasing on $[a,b]$.

If V is a total variation on $[a,b]$, then by thm: 1

$f \in R(V)$ on $[a,b]$.

Then, $f \in R(1 \cdot V + (-1) \cdot \alpha)$ on $[a,b]$ [By thm 2 of unit II]

If assume that the theorem is true for increasing Integrators.

Then $f \in R(V)$ on $[a,b]$ with V is increasing on $[a,b]$

$\Rightarrow f \in R(V)$ on $[c,d]$

Also $f \in R(V-\alpha)$ on $[a,b]$ with $(V-\alpha)$ is increasing on $[a,b]$

$\Rightarrow f \in R(V-\alpha)$ on $[c,d]$

$\therefore f \in R(1 \cdot V + (-1)(V-\alpha))$ on $[c,d]$ [By thm 2 unit 2]

i.e.) $f \in R(V - V + \alpha)$ on $[c,d]$

i.e.) $f \in R(\alpha)$ on $[c,d]$

Hence our aim is to prove that the theorem is true for increasing integrators factors

Increasing functions
(i) It is sufficient to prove the theorem if α is increasing on $[c,d]$

So, let α is increasing on $[a,b]$ and $a < c < d < b$

To prove that: $f \in R(\alpha)$ on $[c,d]$, we have to prove that,

$$\int_c^d f\,d\alpha \text{ exists.}$$

Since $\int_a^c f\,d\alpha + \int_c^d f\,d\alpha = \int_a^d f\,d\alpha$

$\Rightarrow \int_c^d f\,d\alpha = \int_a^d f\,d\alpha - \int_a^c f\,d\alpha$

i.e.) we have to prove that $\int_a^d f\,d\alpha$ and $\int_a^c f\,d\alpha$ exists.

To prove that: $f \in R(\alpha)$ on $[a,d]$ and $f \in R(\alpha)$ on $[a,c]$

To prove that: $f \in R(\alpha)$ on $[a,c]$.

If P is a partition of $[a,x]$

Let $\Delta(P,x)$ denote the difference,

$\Delta(P,x) = U(P,f,\alpha) - L(P,f,\alpha)$ on $[a,x]$

Since $f \in R(\alpha)$ on $[a,b]$. f satisfies Riemann condition w.r.t 'α' on $[a,b]$. then for any given $\epsilon > 0$

$\exists$ a partition P_ϵ of $[a,b]$ such that $\forall$ $P \supseteq P_\epsilon$,

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

i.e.) $\Delta(P,b) < \epsilon$ for every $P \supseteq P_\epsilon$ $\rightarrow (1)$

Assume that $c \in P_\epsilon$

Then the points of P_ϵ in $[a,c]$ from a partition P_ϵ' of $[a,c]$ i.e.) $P_\epsilon' = P_\epsilon \cap [a,c]$

Let P' be a partition of $[a,c]$ finer than P_ϵ'

Then $P = P' \cup P_\epsilon$ is a partition of $[a,b]$ P which is finer

that P_ϵ

then we have

$\Delta(p', a) < \Delta(p, b) \longrightarrow (2)$

for since the points of p consists of all the points

of p' together with p_ϵ in $[c, b]$

Further the sum definining $\Delta(p', \epsilon)$ is a part of

the terms in the sum defining $\Delta(p, b)$ and

part of the sum is ≥ 0.

consider

ie) $\underline{\Delta(p', c)} = U(P', f, \alpha) - L(P', f, \alpha)$

$$= \sum_{k=1}^{n} (M_k(f) - m_k(f)) \Delta\alpha_k$$

$$= (M_1'(f) - m_1'(f))[\alpha(x_1') - \alpha(x_0)]$$

$$+ (M_1''(f) - m_1''(f))[\alpha(x_1) - \alpha(x_1')]$$

$$+ (M_2(f) - m_2(f))[\alpha(x_2) - \alpha(x_1)] +$$

$$\cdots + (M_k(f) - m_k(f))[\alpha(x_k) - \alpha(x_{k-1})]$$

Also,

$$\Delta(P, b) = (M_1'(f) - m_1'(f))[\alpha(x_1') - \alpha(x_0)]$$

$$+ (M_1''(f) - m_1''(f))[\alpha(x_1) - \alpha(x_1')] + \cdots$$

$$+ (M_k(f) - m_k(f))[\alpha(x_k) - \alpha(x_{k-1})] +$$

$$(M_{k+1}(f) - m_{k+1}(f))[\alpha(x_{k+1}) - \alpha(x_k)] + \cdots$$

$$\cdots + (M_n(f) - m_n(f))[\alpha(x_n) - \alpha(x_{n-1})]$$

$$\Rightarrow \Delta(P, b) = \overset{U(P', f, \alpha) - L(P', f, \alpha)}{\cancel{\Delta(P', c)}} + (M_{k+1}(f) - m_{k+1}(f))[\alpha(x_{k+1}) - \alpha(x_k)]$$

$$+ \cdots + (M_n(f) - m_n(f))[\alpha(x_n) - \alpha(x_{n-1})]$$

$U(P, f, \alpha) - L(P, f, \alpha) \geq U(P', f, \alpha) - L(P', f, \alpha)$

$\cancel{\Delta(P', c)} \leq \Delta(P, b)$ $[\because \alpha$ is increasing and

$U(P', f, \alpha) - L(P', f, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha)$ $[M_{k+1}(f) - m_{k+1}(f)] \geq 0$

Using ① & ②

$\Delta(P', c) \le \Delta(P, b) < \epsilon$

$\Rightarrow \Delta(P', c) < \epsilon, \ \forall\ P' \supseteq P'_\epsilon$

Thus for any $\epsilon > 0$, $\exists\ P'_\epsilon$ such that

$\forall\ P' \supseteq P'_\epsilon$ of $[a, c]$ such that

$\Delta(P', c) = U(P', f, \alpha) - L(P', f, \alpha) < \epsilon$

ie) f satisfies Riemann condition w.r.t α on $[a, c]$

$\Rightarrow f \in R(\alpha)$ on $[a, c]$ ie) $\int_a^c f\, d\alpha$ exists

lly by the same argument we can prove that,

$f \in R(\alpha)$ on $[a, d]$. ie) $\int_a^d f\, d\alpha$ exists

Hence $\int_c^d f\, d\alpha = \int_a^d f\, d\alpha - \int_a^c f\, d\alpha$ exists

$\therefore f \in R(\alpha)$ on $[c, d]$.

Theorem: 3

Assume $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ where α increasing on $[a, b]$.

Define $F(x) = \int_a^x f(t)\, d\alpha(t)$ and $G(x) = \int_a^x g(t)\, d\alpha(t)$

If $x \in [a, b]$ then $f \in R(G)$ and $g \in R(F)$ and the product $f \cdot g \in R(\alpha)$ on $[a, b]$. And we have,

$$\int_a^b f(x)\, g(x)\, d\alpha(x) = \int_a^b f(x)\, dG(x) = \int_a^b g(x)\, dF(x)$$

Proof: Given: 1) $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$

(ii) α increasing on $[a, b]$

Since $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ and α increasing on $[a, b]$,

$f \cdot g \in R(\alpha)$ on $[a, b]$ [By Thm : 17 of unit 2]

$\Rightarrow \int_a^b f(x)\, g(x)\, d\alpha(x)$ exists

Define $F(x) = \int_a^x f(t)\, d\alpha(t)$

and $G(x) = \int_a^x g(t)\, d\alpha(t)$

If $x \in [a, b]$

To prove that : $f \in R(G)$ and $\int_a^b f\, dG = \int_a^b f\, d\alpha$

(ie) To prove that,

For any given $\epsilon > 0$, there exists a partition P_ϵ,

Such that $\forall\ P \supseteq P_\epsilon$,

$$|S(P, f, G) - A| < \epsilon$$

For any partition $P = \{x_0, x_1, \dots x_n\}$ of $[a, b]$,

we have,

$$S(P, f, G) = \sum_{k=1}^{n} f(t_k)\, \Delta G_k \quad , \forall\ t_k \in [x_{k-1}, x_k]$$

$$= \sum_{k=1}^{n} f(t_k)\, [G(x_k) - G(x_{k-1})]$$

$$= \sum_{k=1}^{n} f(t_k)\, \left[\int_a^{x_k} g(t)\, d\alpha(t) - \int_a^{x_{k-1}} g(t)\, d\alpha(t) \right]$$

$$= \sum_{k=1}^{n} f(t_k) \left[\int_a^{x_k} g(t)\, d\alpha(t) + \int_{x_{k-1}}^{a} g(t)\, d\alpha(t) \right] \quad \text{[given]}$$

$$= \sum_{k=1}^{n} f(t_k) \int_{x_{k-1}}^{x_k} g(t)\, d\alpha(t)$$

$$\therefore S(P, f, G) = \sum_{k=1}^{n} f(t_k) \int_{x_{k-1}}^{x_k} g(t)\, d\alpha(t) \longrightarrow (1)$$

Let $A = \int_a^b f(x)\, g(x)\, d\alpha(x)$